Confidence Intervals

Data Analysis for Psychology in R 1

Semester 2, Week 1

Dr Umberto Noè

Department of Psychology
The University of Edinburgh

Learning objectives

1. Understand the importance of a confidence interval.

2. Understand the link between standard errors and confidence intervals.

3. Understand how to construct a confidence interval for an unknown parameter of interest.

Part A

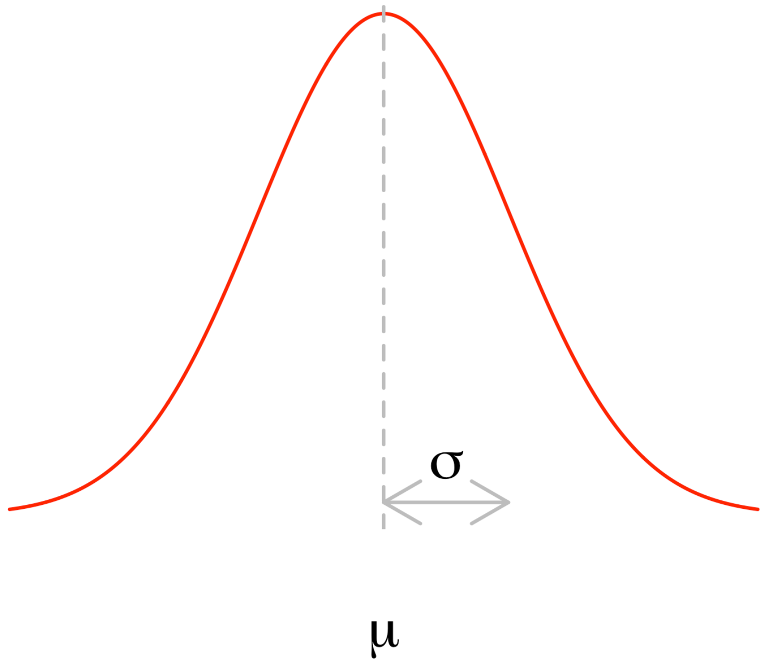
Recap

Normal distribution

- Variable = any characteristic of the units under investigation.
- A variable X follows a Normal distribution with mean μ and standard deviation σ if its distribution is bell-shaped and symmetric.
- Using mathematical symbols, we write that X follows a Normal distribution with mean μ and standard deviation σ as:

$$X \sim N(\mu, \sigma)$$

- μ specifies the centre of the distribution
- σ specifies the spread of the distribution



Normal distribution: probability vs quantile

Find the probability p to the LEFT of a value x:

```
p <- pnorm(x, mean = mu, sd = sigma)
```

Find the value x having a probability p to its LEFT:

```
x <- qnorm(p, mean = mu, sd = sigma)
```

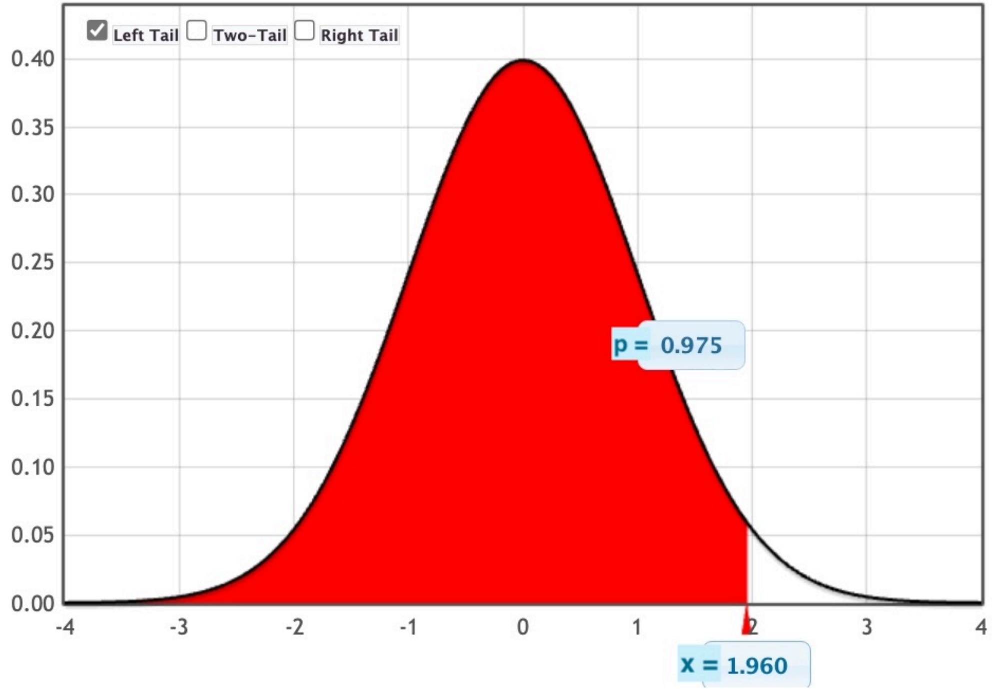
Example with $N(0, 1)$:

```
qnorm(0.975)
```

```
## [1] 1.96
```

```
pnorm(1.96)
```

```
## [1] 0.975
```



Standardisation / z-scoring

- Let $X \sim N(\mu, \sigma)$. *In words: Let X follow a normal distribution with mean μ and standard deviation σ.*
- Define:

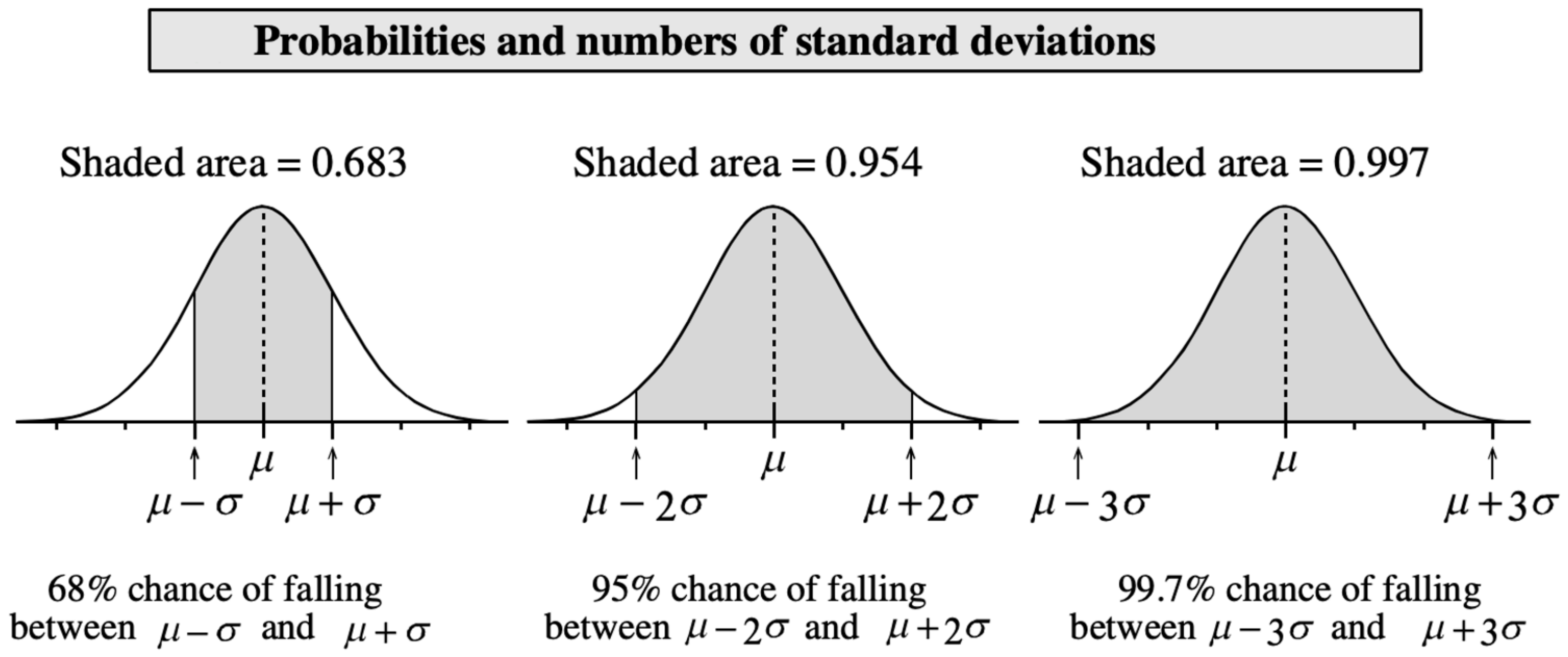
$$Z = \frac{X - \mu}{\sigma}$$

- $Z \sim N(0, 1)$. *In words: Z follows a standard normal distribution.*
  - $\mu_Z = 0$
  - $\sigma_Z = 1$
- To transform Z back to X we use this transformation:

$$X = \mu + Z \cdot \sigma$$

Normal 68–95–99.7 rule

- Recall that for a random variable $X \sim N(\mu, \sigma)$, **roughly** 95% of the values fall between $\mu - 2\sigma$ and $\mu + 2\sigma$:



Normal 68–95–99.7 rule

- The interval below contains **roughly** 95% of the values in the distribution:

$$[\mu - 2 \cdot \sigma,\ \mu + 2 \cdot \sigma]$$

- To be more accurate, we need to find the x-values (quantiles) that have 0.025 probability to the left and 0.025 probability to the right, leaving 0.95 probability in the middle.

```
qnorm(c(0.025, 0.975))    # using a N(0,1)
```

```
## [1] -1.96  1.96
```

```
qnorm(0.025) #or qnorm(0.025, lower.tail=TRUE)
```

```
## [1] -1.96
```

```
qnorm(0.025, lower.tail = FALSE)
```

```
## [1] 1.96
```

Normal 68–95–99.7 rule

- The values -1.96 and 1.96 are the quantiles of a standard Normal distribution, cutting a probability of 0.025 in each of the two tails of the distribution.
- To have the quantiles for the original variable $X \sim N(\mu, \sigma)$ we need to transform Z back to X with the formula previously mentioned $(x = \mu + z \cdot \sigma)$:

$$\begin{aligned} z = -1.96 \quad &\rightarrow \quad x = \mu - 1.96 \cdot \sigma \\ z = 1.96 \quad &\rightarrow \quad x = \mu + 1.96 \cdot \sigma \end{aligned}$$

- The interval comprising exactly 95% of the values of X is the range of values *from* $\mu - 1.96 \cdot \sigma$ *to* $\mu + 1.96 \cdot \sigma$, which in mathematics is written as:

$$[\mu - 1.96 \cdot \sigma,\ \mu + 1.96 \cdot \sigma]$$

Estimation

- Without loss of generality, we will focus on the mean as the numerical summary of data.

$$\begin{array}{cccccc} \text{Population mean, } \mu & \rightarrow & \text{unknown} & \rightarrow & \text{example of a parameter} \\ \text{Sample mean, } \bar{x} & \rightarrow & \text{we can compute it} & \rightarrow & \text{example of a statistic} \end{array}$$

- We are typically interested in **estimating an unknown population mean** μ (a **parameter**) using the **mean computed on a random sample** $\bar{x}$ (a **statistic**).
    - We will equivalently call the statistic (sample mean) the **estimate**.
- When estimating an unknown parameter, we should report both

  a) the estimate;
  b) a measure of our "uncertainty" in the estimate.

Some facts

- Statistics vary from sample to sample and have a **sampling distribution**.
- The standard deviation of the sampling distribution is called the **standard error** (SE)
- **Informally**: SE tells us the size of the typical "estimation error" (= $\bar{x} - \mu$).
- $SE = SE_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$

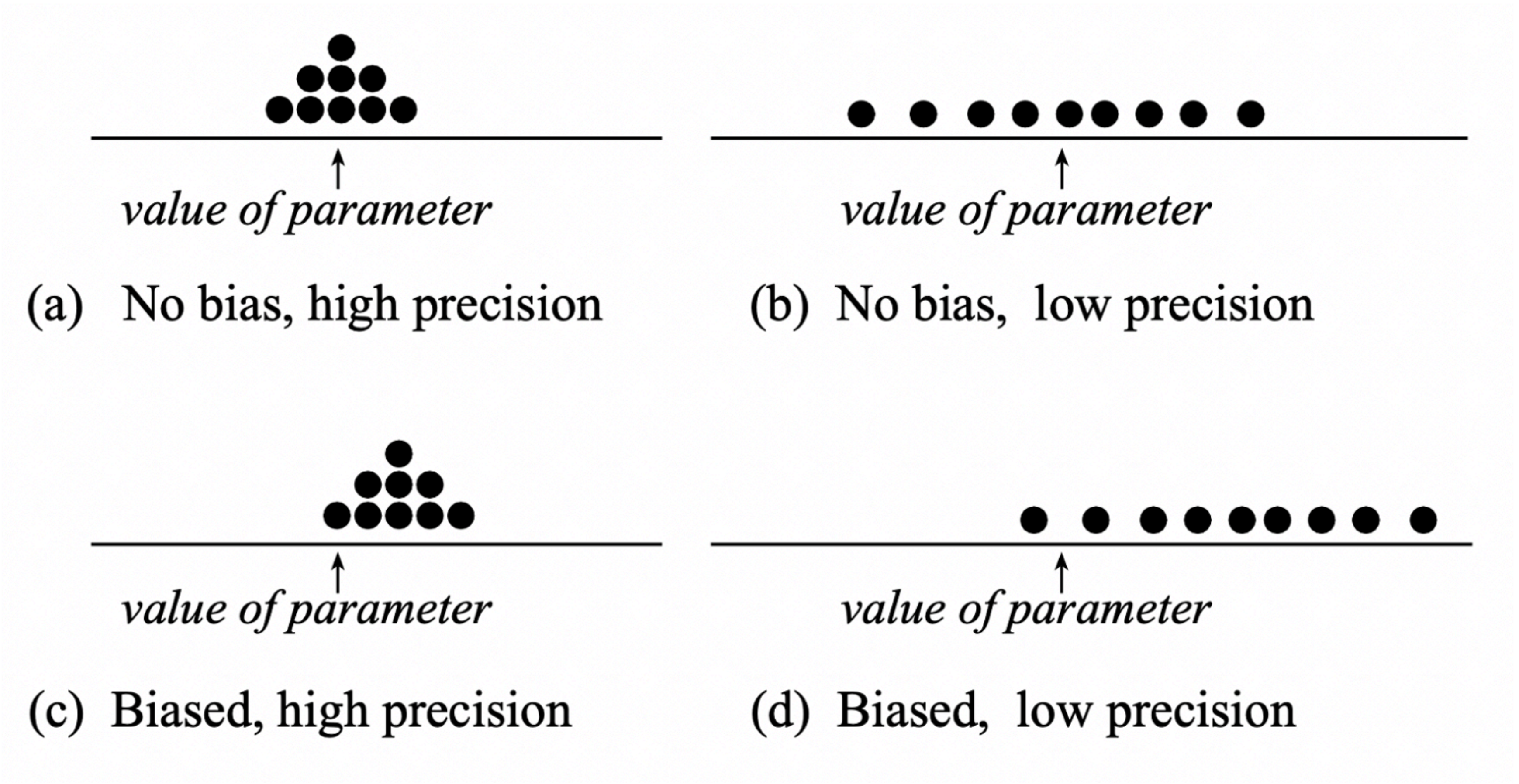
Estimation

- The estimate for a population mean is the sample mean, $\bar{x}$.
- Let's now turn to the key question of reporting uncertainty in the estimate:

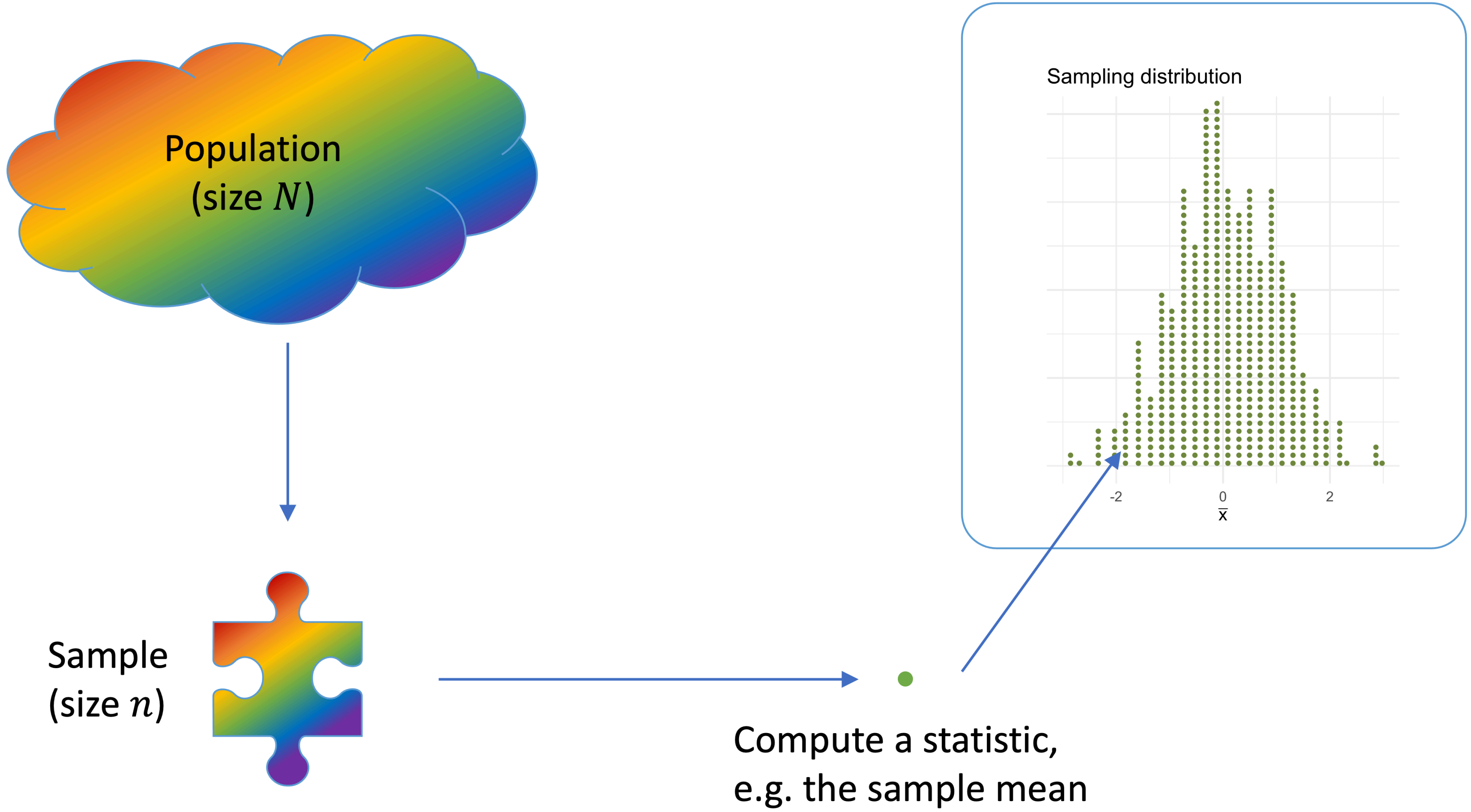
> How accurate is our estimate?

- In other words, how accurate is our statistic $\bar{x}$ as an estimate of the unknown parameter μ?
- Accuracy is a combination of two things:
    - No bias
    - Precision
- We avoid bias if we use random sampling. We have bias if our samples systematically do not include a part of the population.
    - If you choose convenience samples, you will systematically over-estimate or under-estimate the true value.
- Precision relates to the variability of the sampling distribution, and the Standard Error (SE) is used to quantify precision.
    - As the SE gets smaller, the sample means will tend to be closer to the population mean

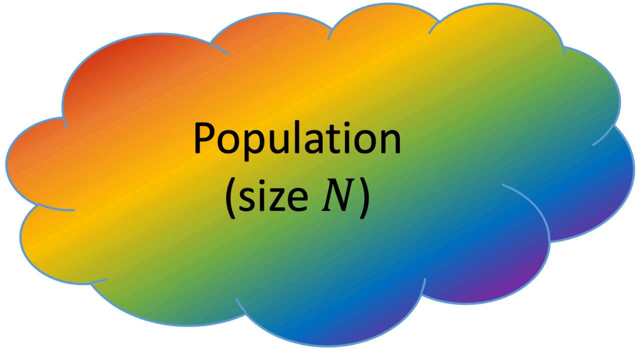
Bias vs Precision



Sampling distribution

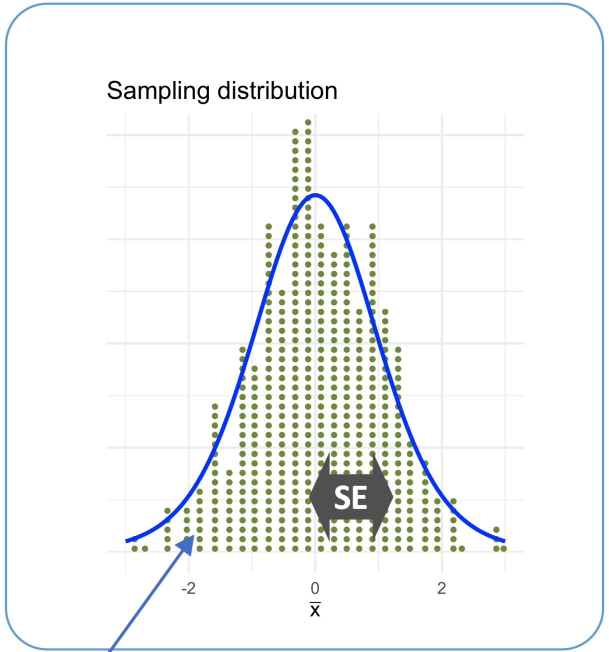


Sampling distribution

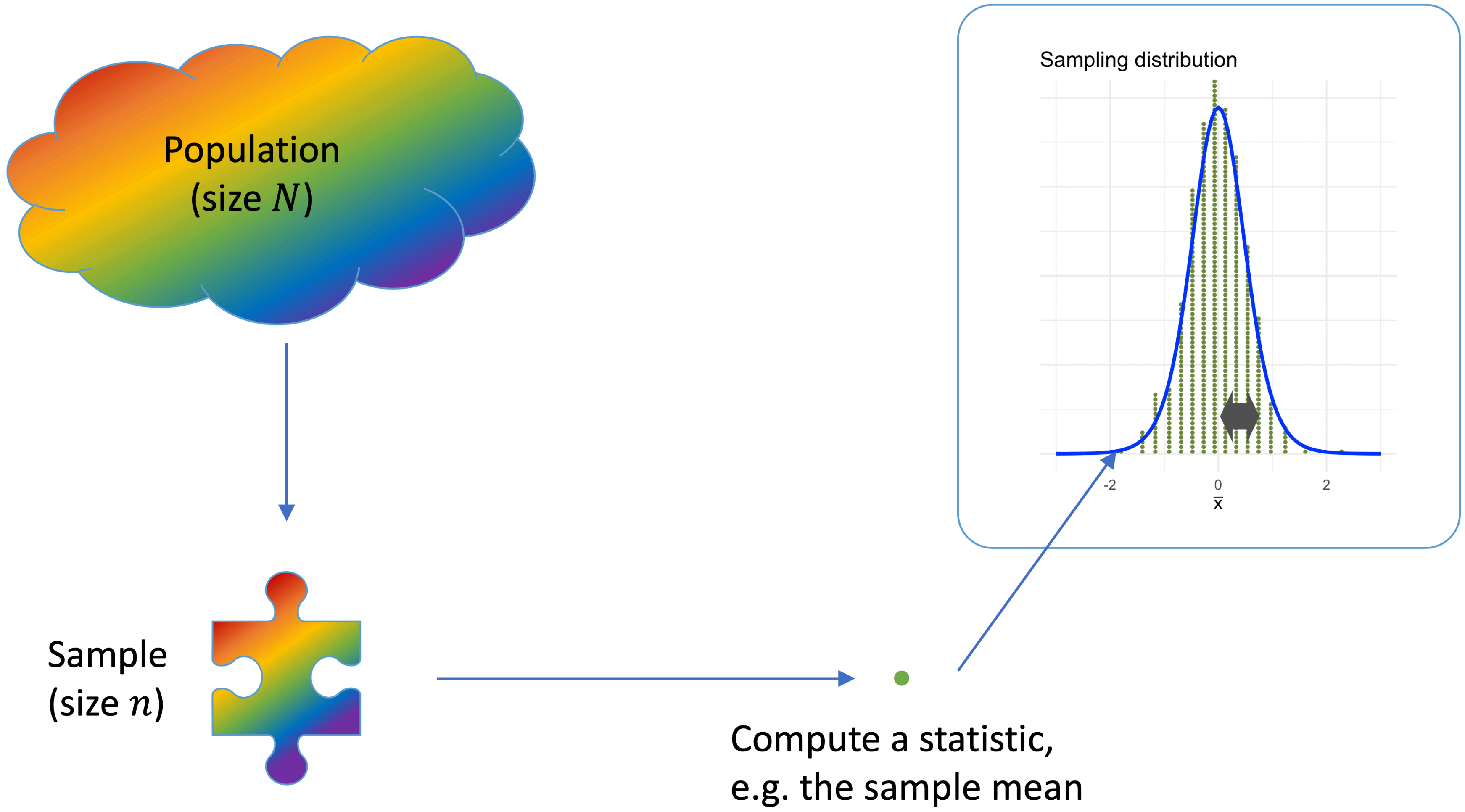


Sample
(size n)

Compute a statistic,
e.g. the sample mean



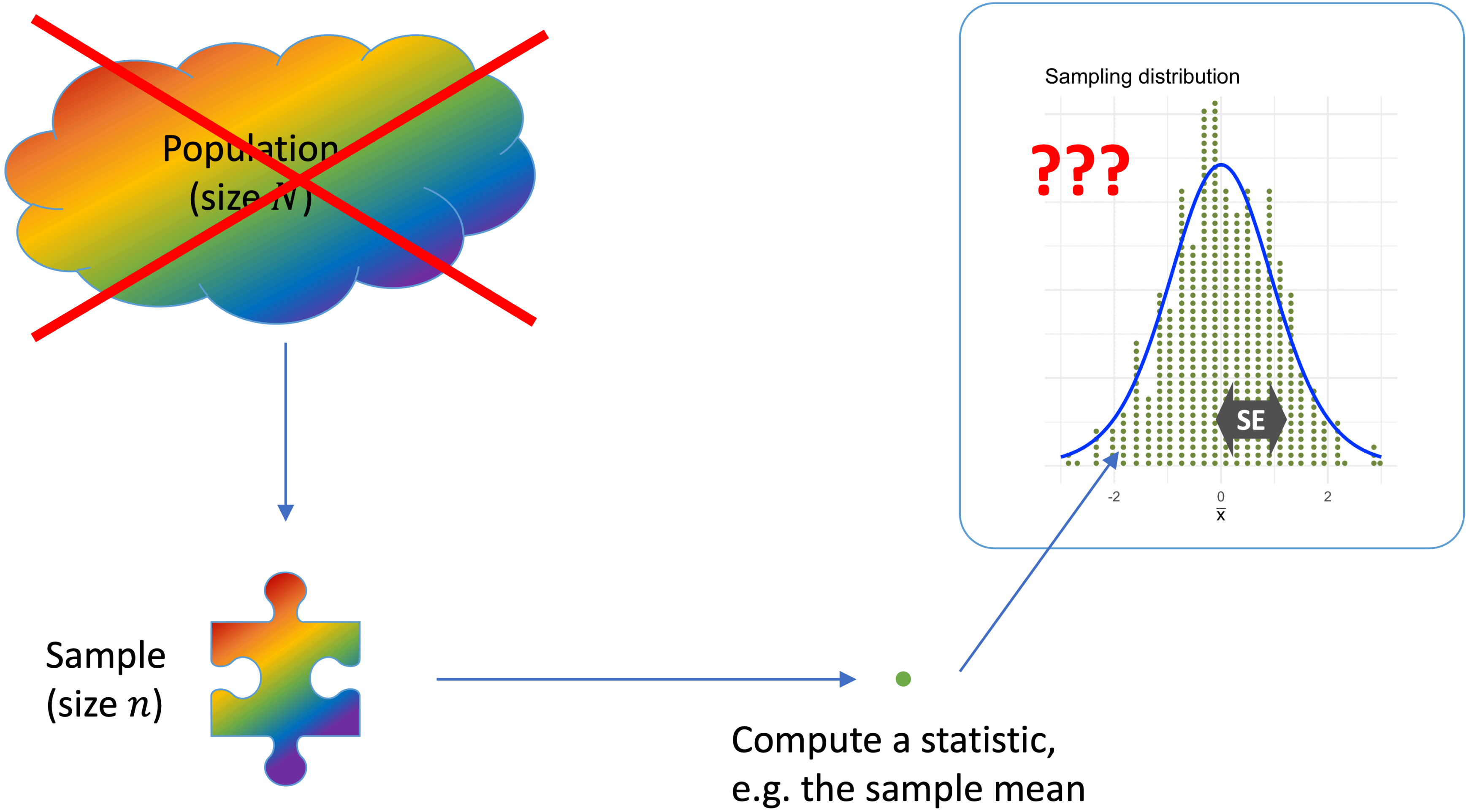
Sampling distribution



Part B

One sample only

One sample only



One sample only: Precision of sample mean

- If we do NOT have the population data:
  - we cannot compute μ, the population mean
  - we also cannot compute σ, the population standard deviation
- Recall that σ is required to assess the precision of the sample mean by computing the SE:

$$SE = \frac{\sigma}{\sqrt{n}}$$

- How can we compute the SE of the mean if we **do not have data on the full population**, and we **can only afford one sample** of size n?

One sample only: Precision of sample mean

- We must also estimate σ with the corresponding sample statistic.
- Substitute σ with the standard deviation computed in the sample, s.
- Standard error of the mean becomes:

$$SE = \frac{s}{\sqrt{n}}$$

- Report estimate (sample mean), along with a measure of its precision (the above SE).

Part C

Confidence Intervals

Key idea

- Parameter estimate = single number. Almost surely the true value will be different from our estimate.
- Range of plausible values for the parameter, called **confidence interval**. More likely that the true value will be captured by a range.

Point estimate	Confidence interval
	

Confidence interval

- Confidence interval (CI) = range of plausible values for the parameter.
- To create a confidence interval we must decide on a confidence level.
- Confidence level = a number between 0 and 1 specified by us. How confident do you want to be that the confidence interval will contain the true parameter value?
- The larger the confidence level, the wider the confidence interval.
    - How confident are you that I am between 39 and 42 years old?
    - How confident are you that I am between 35 and 50 years old?
    - How confident are you that I am between 18 and 70 years old?
- Typical confidence levels are 90%, 95%, and 99%.

CI for the population mean

- Recall that if $X \sim N(\mu, \sigma)$, 95% of the values are between

$$[\mu - 1.96 \cdot \sigma,\ \mu + 1.96 \cdot \sigma]$$

- The sample mean follows a normal distribution:

$$\bar{X} \sim N(\mu_{\overline{X}}, \sigma_{\overline{X}})$$

where:

- $\mu_{\overline{X}} = \mu$
- $\sigma_{\overline{X}} = SE = \dfrac{\sigma}{\sqrt{n}}$

- Substitute in the interval above:

$$[\mu_{\bar{X}} - 1.96 \cdot \sigma_{\bar{X}},\ \mu_{\bar{X}} + 1.96 \cdot \sigma_{\bar{X}}]$$

- That is:

$$\left[\mu - 1.96 \cdot \frac{\sigma}{\sqrt{n}},\ \mu + 1.96 \cdot \frac{\sigma}{\sqrt{n}}\right]$$

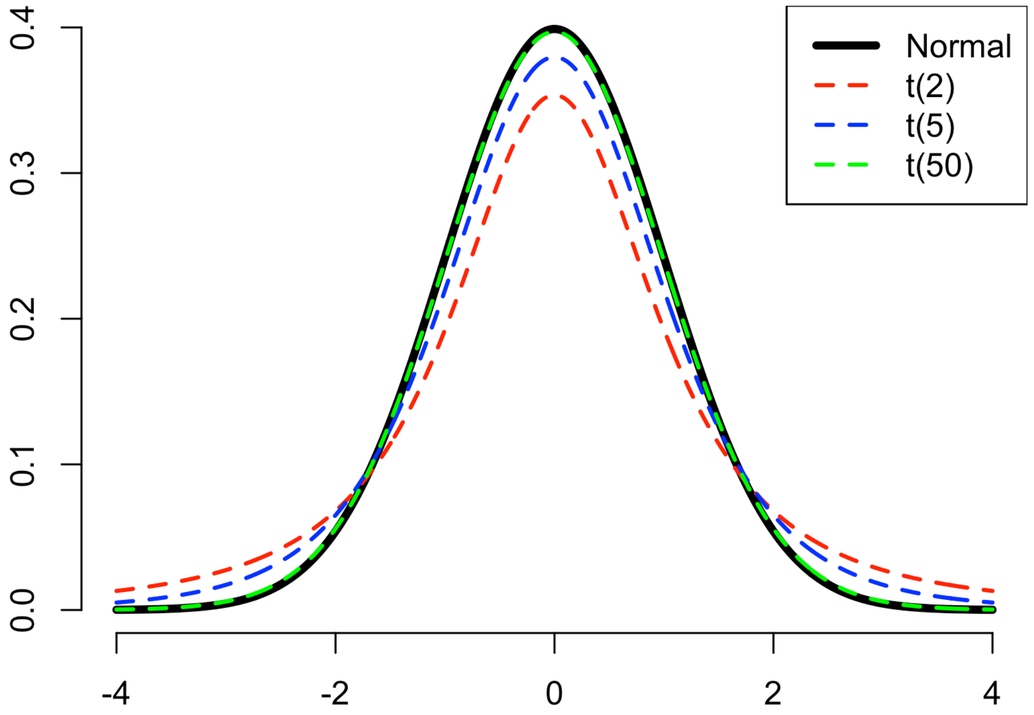
Estimates of μ and σ

- Recall that we do not have the full population data. **We can only afford one sample!**
- We don't have the population mean μ and we estimated it with the sample mean $\bar{x}$
- However, we also don't have σ so we need to estimate it with s, the sample standard deviation:

$$\left[\bar{x} - 1.96 \cdot \frac{s}{\sqrt{n}},\ \bar{x} + 1.96 \cdot \frac{s}{\sqrt{n}}\right]$$

- However, this interval is now **wrong!**
- Because we didn't know σ and we had to estimate it with s, this bring and **extra element of uncertainty**
- As we are unsure about the actual value of the population standard deviation, the reference distribution is no longer Normal, but a distribution that is more "uncertain" and places higher probability in the tails of the distribution.
- When the population standard deviation is unknown, the sample mean follows a t-distribution.
- The quantiles -1.96 and 1.96 refer to the normal distribution, so these are wrong and we need to find the correct ones!

t-distribution



t-distribution

- A distribution similar to the standard Normal distribution, also with a zero mean
- Depends on a number called **degrees of freedom** (DF) = sample size - 1. That is, $df = n - 1$.
- We write the distribution as:

$$t(n-1)$$

- Suppose the sample size is 20. In R:

```
qt(0.025, df = 19)     # quantile = t-value with 0.025 prob to the LEFT
```

```
## [1] -2.093
```

```
pt(-2.093, df = 19)    # prob to the LEFT of t = -2.093
```

```
## [1] 0.025
```

Finally: the correct confidence interval

- Now we can finally compute the correct confidence interval.
- We need to replace the quantiles with those from the $t(n-1)$ distribution, denote them by $-t^*$ and $+t^*$, and these will be different all the time as they depend on the sample size.
- Generic form the of the CI for the mean:

$$\left[\bar{x} - t^* \cdot \frac{s}{\sqrt{n}},\ \bar{x} + t^* \cdot \frac{s}{\sqrt{n}}\right]$$

- Generic form the of the 95% CI for the mean with a sample of size $n = 20$:

```r
qt(c(0.025, 0.975), df = 20 - 1)
```

```
## [1] -2.093  2.093
```

$$\left[\bar{x} - 2.093 \cdot \frac{s}{\sqrt{n}},\ \bar{x} + 2.093 \cdot \frac{s}{\sqrt{n}}\right]$$

Other confidence levels

- Generic form the of the 99% CI for the mean with a sample of size $n = 20$:

```r
qt(c(0.005, 0.995), df = 20 - 1)
```

```
## [1] -2.861  2.861
```

$$\left[\bar{x} - 2.861 \cdot \frac{s}{\sqrt{n}},\ \bar{x} + 2.861 \cdot \frac{s}{\sqrt{n}}\right]$$

Example: 95% CI for the pop. mean salary

- Parameter of interest: mean yearly salary of a NFL player in the year 2019, denoted μ.
- Sample of 50 players:

```
library(tidyverse)
nfl_sample <- read_csv("https://uoepsy.github.io/data/NFLSample2019.csv")
dim(nfl_sample)
```

```
## [1] 50  5
```

```
head(nfl_sample)
```

```
## # A tibble: 6 × 5
##   Player          Position Team     TotalMoney YearlySalary
##   <chr>           <chr>    <chr>         <dbl>        <dbl>
## 1 Najee Goode     43OLB    Jaguars       0.805        0.805
## 2 Jack Crawford   43DT     Falcons       9.9          2.48
## 3 Tra Carson      RB       Lions         1.23         0.615
## 4 Jordan Richards S        Ravens        0.805        0.805
## 5 Desmond Trufant CB       Falcons      68.8         13.8
## 6 Alex Anzalone   43OLB    Saints        3.47         0.866
```

Example: 95% CI for the pop. mean salary

```
xbar <- mean(nfl_sample$YearlySalary)
xbar
```

```
## [1] 3.359
```

```
s <- sd(nfl_sample$YearlySalary)
s
```

```
## [1] 4.312
```

```
n <- nrow(nfl_sample)
n
```

```
## [1] 50
```

```
SE <- s / sqrt(n)
SE
```

```
## [1] 0.6098
```

```
tstar <- qt(c(0.025, 0.975), df = n-1)
tstar
```

```
## [1] -2.01  2.01
```

```
xbar - 2.01 * SE
```

```
## [1] 2.133
```

```
xbar + 2.01 * SE
```

```
## [1] 4.584
```

or:

```
xbar + tstar * SE
```

```
## [1] 2.133 4.584
```

Example: 95% CI for the pop. mean salary

- The 95% confidence interval for the mean salary of **all** NFL players in the year 2019 is [2.13, 4.58] million dollars.
- Write this up as:

> We are 95% confident that the average salary of a NFL player in 2019 was between 2.13 and 4.58 million dollars.

- If it makes more sense in your sentence, you can report the sample mean followed by the CI in brackets (to tell the reader how precise your estimate is).
- Use the format M = ..., 95% CI [..., ...].

Part D

Warning on interpretation

Warning!

- If you had many random samples and computed a 95% confidence interval from each sample:
  - about 95% of those intervals will contain the true parameter value
  - about 5% of those intervals will **not** contain the true parameter value
- Example 1: if you had 100 random samples and computed a 95% confidence interval from each sample:
  - about 95 (= 100 * 0.95) of those intervals will contain the true parameter value
  - about 5 (= 100 * 0.05) of those intervals will **not** contain the true parameter value
- Example 2: if you had 20 random samples and computed a 95% confidence interval from each sample:
  - about 19 (= 20 * 0.95) of those intervals will contain the true parameter value
  - about 1 (= 20 * 0.05) of those intervals will **not** contain the true parameter value

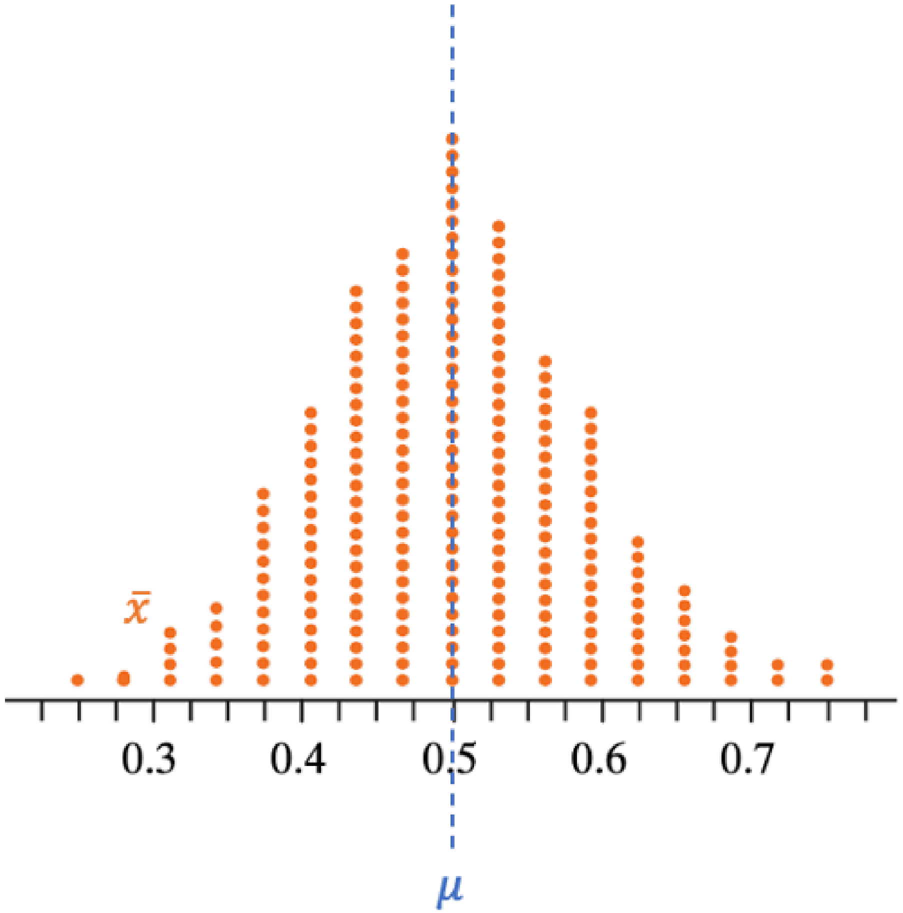
Warning!

- Consider again example 2, where you have 20 random samples and built a confidence interval from each sample.
- We speak about **probability** when we refer to the **collection** of those 20 confidence intervals.
  That is, the probability the that **collection** of confidence intervals will contain the true parameter value is 0.95.
  - Think of this as

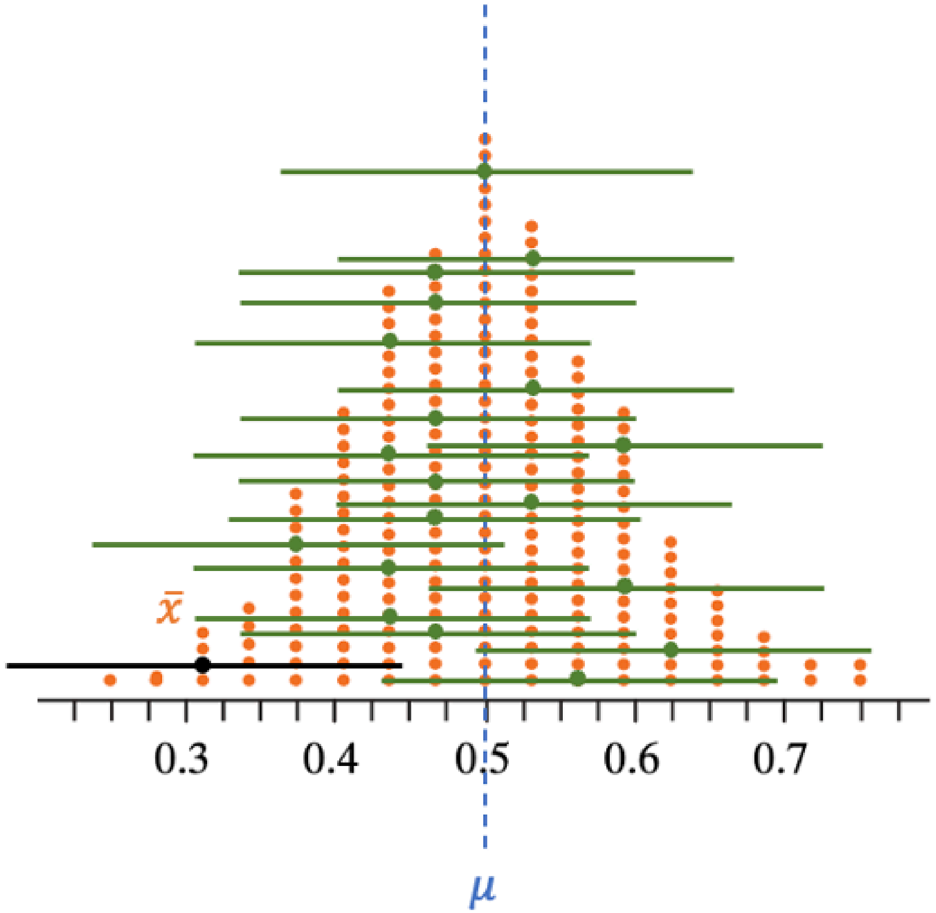
$$\frac{\text{number of CIs containing } \mu}{\text{total number of CIs}} = \frac{19}{20} = 0.95$$

- We speak of **confidence** when we refer to just **one** confidence interval that we have computed.
  Say the 95% CI is [2.5, 5.3] min. We would say: we are 95% confident that the population mean is between 2.5 and 5.3 minutes.
  - It is **wrong** to say that there is a 95% probability that the population mean is between 2.5 and 5.3 minutes.

Warning!



Warning!



This week

Tasks

- Attend both lectures
- Attend your lab and work together on the lab tasks
  - Tip: read the worked example in advance!
- Complete any lecture activities and/or readings
- Complete the weekly quiz
  - Opens Monday at 9am
  - Closes Sunday at 5pm

Support

- **Office hours**: for one-to-one support on course materials or assessments
  (see LEARN > Course information > Course contacts)
- **Piazza**: help each other on this peer-to-peer discussion forum
- **Student Adviser**: for general support while you are at university
  (find your student adviser on MyEd/Euclid)