Chi-square Tests

Data Analysis for Psychology in R 1

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Weeks Learning Objectives

1. Understand how to perform a χ^2 goodness-of-fit and interpret the results.
2. Understand how to perform a χ^2 test of independence and interpret the results.
3. Conduct and interpret the assumption checks for χ^2 tests.

Topics for today

- Recording 1:
    - Types of χ^2 test
    - Worked example of χ^2 goodness-of-fit
    - Relative, observed and expected frequencies
- Recording 2:
    - Worked example of χ^2 goodness-of-fit
    - Inferential testing, and write up.
- Recording 3:
    - Worked example of χ^2 test of independence.
- Recording 4:
    - Residuals, assumptions and effect size measures.
- Bonus slides: For those who are interested, the full calculations for recording 2 are given in slides.

Purpose

- χ^2 goodness of fit test
  - The primary purpose is to test whether the collected data (observed frequencies) are consistent with a hypothesized/known distribution (expected frequencies).
- χ^2 test of independence:
  - We have 2 categorical variables, drawn from a single population.
  - We want to know if the variables are independent or not.
  - If the category membership is dependent, then knowing what category someone is in on variable 1, helps us predict what category they would be in for variable 2.

Data Requirements

- χ^2 goodness of fit test
  - Single categorical variable
- χ^2 test of independence:
  - Two categorical variables.

Example: Goodness of fit

- Suppose we are interested in the distribution of students across three final year psychology options (Social, Differential, Developmental).
- We have data from 2014-15, and we want to know if the distribution is the same in 2015-16.

Data

```
head(class)
```

```
## # A tibble: 6 × 2
##   ID    course
##   <chr> <fct>
## 1 ID1   Differential
## 2 ID2   Social
## 3 ID3   Social
## 4 ID4   Social
## 5 ID5   Social
## 6 ID6   Developmental
```

- `ID` = Unique ID variable
- `course` = factor with 3 levels (Social, Differential, Developmental)

Observed frequencies

```
tab1 <- class %>%
  group_by(course) %>%
  tally()
```

```
tab1
```

```
## # A tibble: 3 × 2
##   course            n
##   <fct>         <int>
## 1 Differential     28
## 2 Social           62
## 3 Developmental    60
```

Relative frequencies

- In 2014-15, the department had the following proportions:
  - Social = 0.50, or 50%
  - Differential = 0.30, or 30%
  - Developmental = 0.20, or 20%

Relative frequencies

```
tab1 <- tab1 %>%
  transmute(
    course = course,
    relative = c(0.30, 0.50, 0.20),
    observed = n
  )
```

```
tab1
```

```
## # A tibble: 3 × 3
##   course        relative observed
##   <fct>            <dbl>    <int>
## 1 Differential       0.3       28
## 2 Social             0.5       62
## 3 Developmental      0.2       60
```

Expected frequencies

- Given this, and a total number of students (n=150) for the current year, we can calculate the expected frequencies for each area.
    - $Expected = Relative * N$

Put it together

```
tab1 <- tab1 %>%
  mutate(
    expected = relative*sum(observed)
  )
```

```
tab1
```

```
## # A tibble: 3 × 4
##   course        relative observed expected
##   <fct>            <dbl>    <int>    <dbl>
## 1 Differential       0.3       28       45
## 2 Social             0.5       62       75
## 3 Developmental      0.2       60       30
```

Time for a break

Welcome Back!

Now we have discussed how to calculate the core values from our data, let's think about our hypotheses, test statistic, and inferential testing.

Hypotheses

$$H_0 = P(0.20, 0.50, 0.30)$$
$$H_1 \neq P(0.20, 0.50, 0.30)$$

- H_0 says that the data follow a specific and known pattern or probabilities (frequencies)
- H_1 says they don't

Test statistic

$$\chi^2 = \sum_{i=1}^{k} \frac{(E_i - O_i)^2}{E_i}$$

- E_i = expected frequencies
- O_i = observed frequencies
- $\sum_{i=1}^{k}$ = do the calculation starting from cell 1 through to cell k (k=number groups) and add them up.

Null Distribution

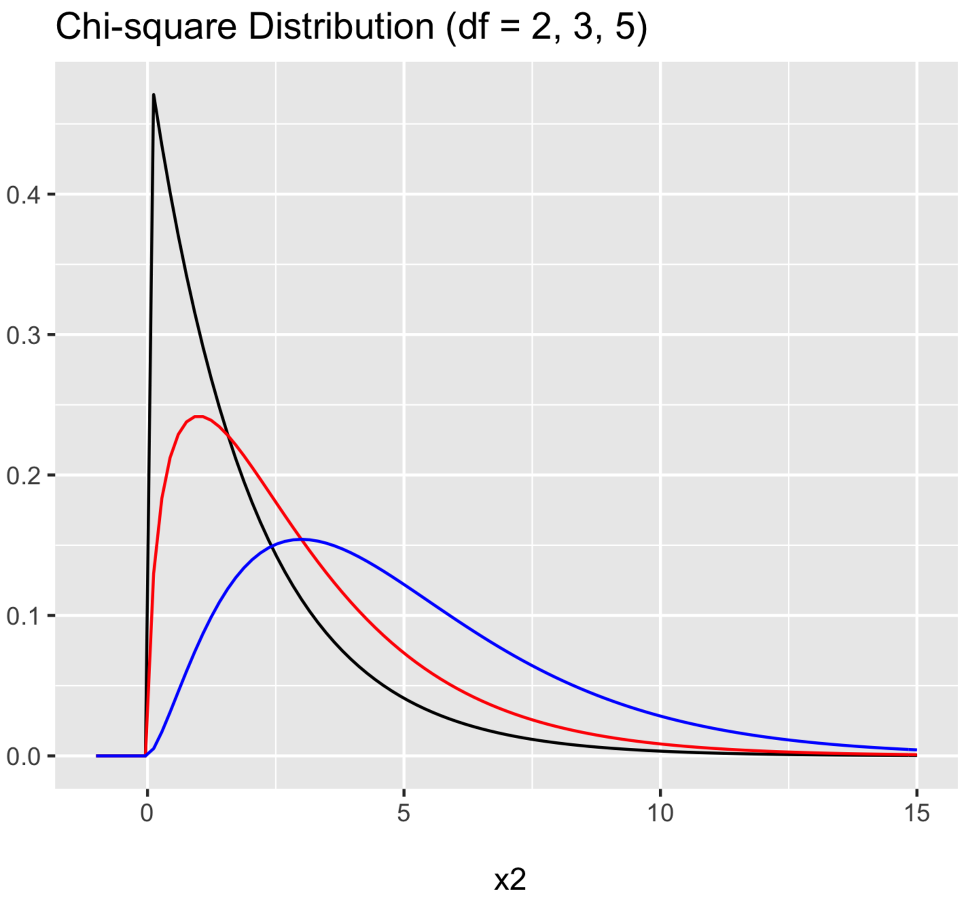
- Sampling distribution for χ^2 test is a χ^2 distribution.
- χ^2 distribution describes the distribution of the sum of k squared independent standard normal variables.
    - Huh?

$$\chi^2 = \sum_{i=1}^{k} \frac{(E_i - O_i)^2}{E_i}$$

Null Distribution

- Parameter of the χ^2 distribution is degrees of freedom (df)
    - Just like t-test.
- df are determined by the number of categories (k)
- Goodness of fit test has $k-1$ degrees of freedom.
    - Why?

Null Distribution



- The plot shows χ^2 distributions for 2 (black), 3 (red), and 5 (blue) df's
- Note that as the df increase, the area under the curve for smaller values increases.
- What does that mean?
  - It means as we add up more things, we would expect the random fluctuations from 0 to to also increase.
  - In any given sample, even if the null is true in the population, sampling variability would mean we have some non-zero values.
  - So we need to account for this.

Calculation

```
tab1 <- tab1 %>%
  mutate(
    step1 = expected - observed,
    step2 = step1^2,
    step3 = step2/expected
  )
tab1
```

```
## # A tibble: 3 × 7
##   course        relative observed expected step1 step2 step3
##   <fct>            <dbl>    <int>    <dbl> <dbl> <dbl> <dbl>
## 1 Differential       0.3       28       45    17   289  6.42
## 2 Social             0.5       62       75    13   169  2.25
## 3 Developmental      0.2       60       30   -30   900 30
```

- Step1 = $E_i - O_i$
- Step2 = $(E_i - O_i)^2$
- Step3 = $\frac{(E_i - O_i)^2}{E_i}$

Calculation

- Last step is to sum the values for step 3 to get the χ^2

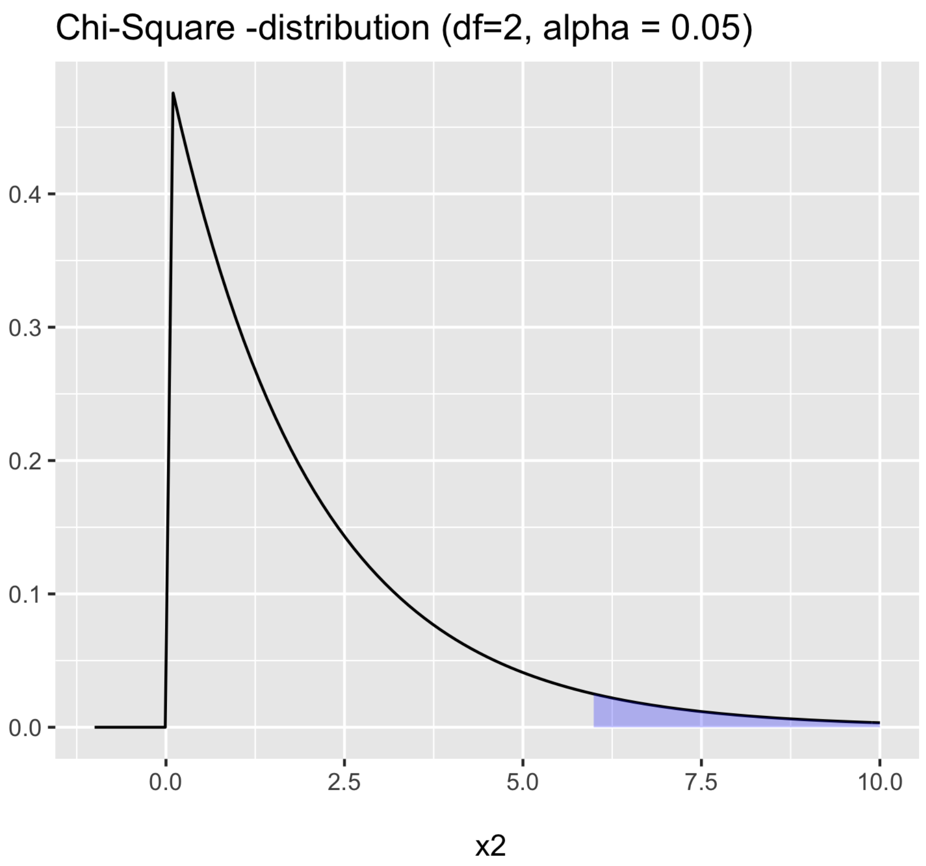
```
x2 <- sum(tab1$step3)
x2
```

```
## [1] 38.67556
```

Is my test significant?

- χ^2 = 38.68
- Degrees of freedom = 3-1 = 2
- α = 0.05

Is my test significant?



Is my test significant?

```
tibble(
  CritValue = round(qchisq(0.95, 2),2),
  Exactp = round(1-pchisq(x2, 2),5)
)
```

```
## # A tibble: 1 × 2
##   CritValue Exactp
##       <dbl>  <dbl>
## 1      5.99      0
```

In R

```r
gof_res <- chisq.test(tab1$observed, p = c(0.3, 0.5, 0.2))
gof_res
```

```
##
##      Chi-squared test for given probabilities
##
## data:  tab1$observed
## X-squared = 38.676, df = 2, p-value = 3.997e-09
```

Write up

A χ^2 goodness of fit test was conducted in order to investigate whether the distribution of students across Social, Developmental and Differential classes was equivalent in 2014- 15 and 2015-16. The goodness of fit test was significant (χ^2(2) = 38.68, p<.05) and thus the null hypothesis was rejected. The distribution of student's across courses differs between the two academic years.

Time for a break

Welcome Back!

We will now follow the same steps for a test of independence.

Example: Independence

- I have conducted an experiment with three conditions (n=120, 40 per group)
- I want to check whether my participants are equally distributed based on some demographic variables.
  - Let's focus on whether English is participants first language
- Recall from an experimental design perspective, I want such things to be randomized across my groups.
  - So I would expect an even distribution.

Data

```
head(exp)
```

```
## # A tibble: 6 × 3
##   ID    condition lang
##   <chr> <chr>     <chr>
## 1 ID1   control   Yes
## 2 ID2   control   No
## 3 ID3   control   No
## 4 ID4   control   Yes
## 5 ID5   control   No
## 6 ID6   control   No
```

- `ID` = Unique ID variable
- `condition` = experimental conditions (control, group1, group2)
- `lang` = binary Yes/No for English as first language

Tabular format

- It can be very useful to display data for two categorical variables as a contingency table.

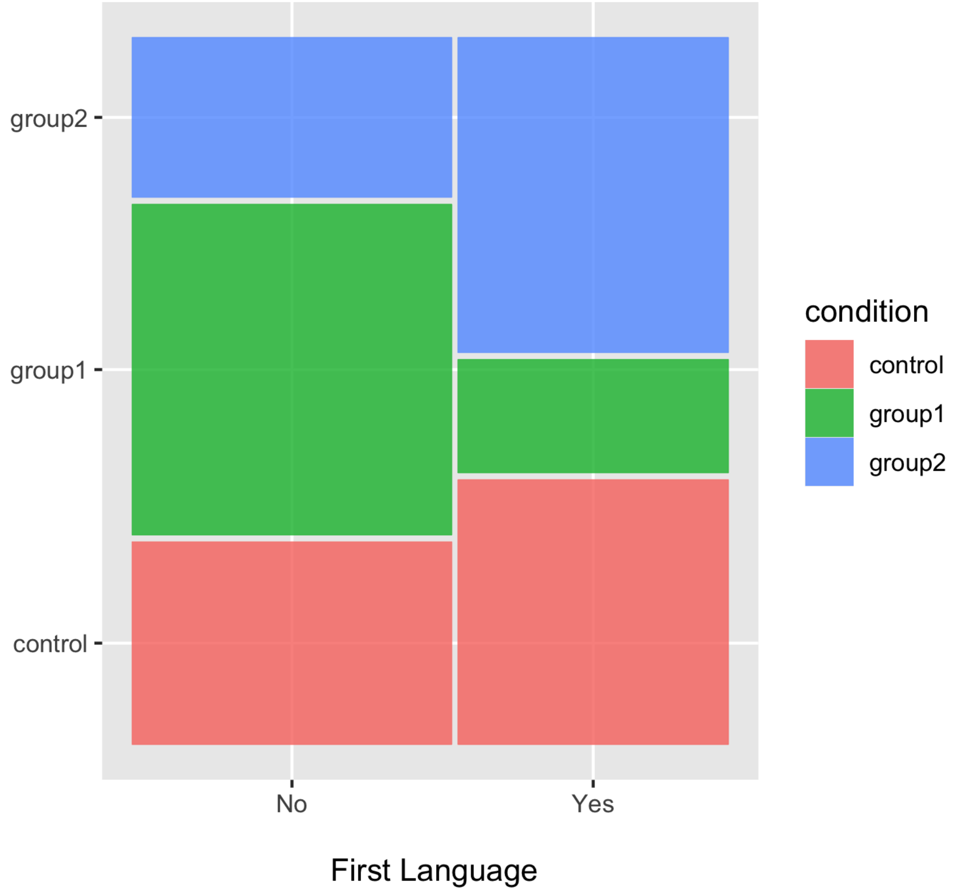
```
tabs <- addmargins(table(exp$condition, exp$lang))
tabs
```

```
##
##              No Yes Sum
##   control    19  21  40
##   group1     31   9  40
##   group2     15  25  40
##   Sum        65  55 120
```

Visualizing Data: Mosaic Plot

```
#install.packages("ggmosaic")
#library(ggmosaic)

ggplot(data = exp) +
  geom_mosaic(aes(x=product(condition, lang),
                  fill = condition)) +
  labs(x = "\n First Language", y = "")
```



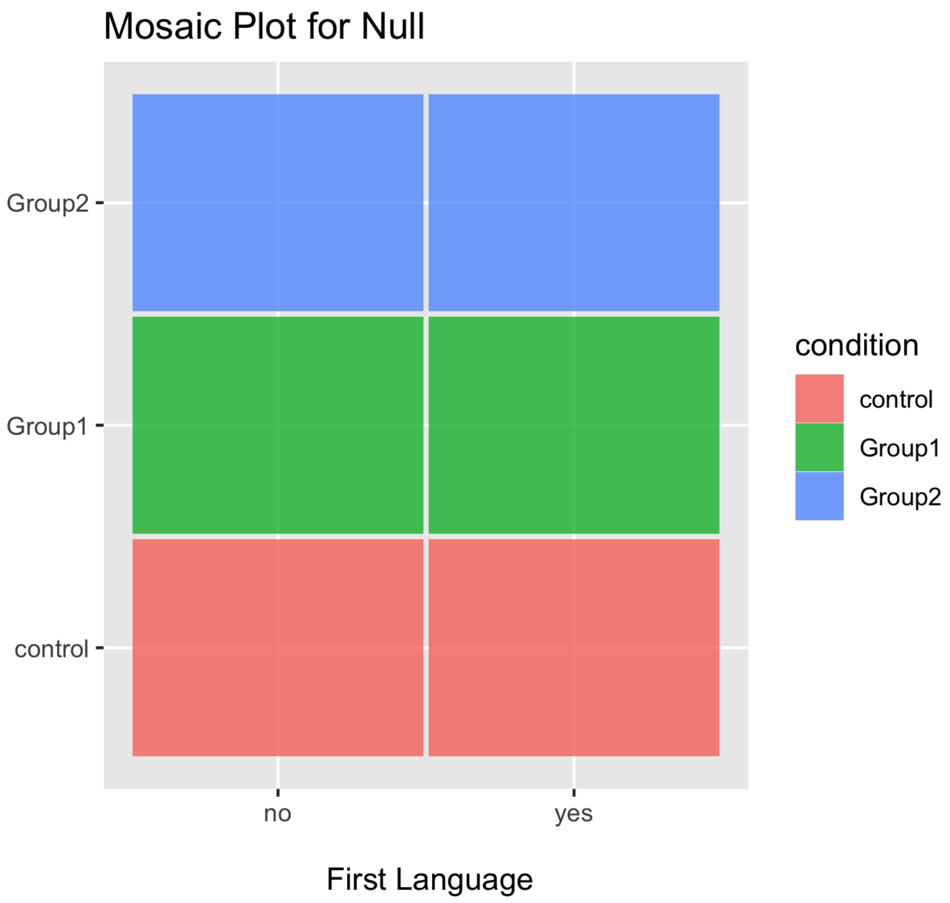
Hypotheses

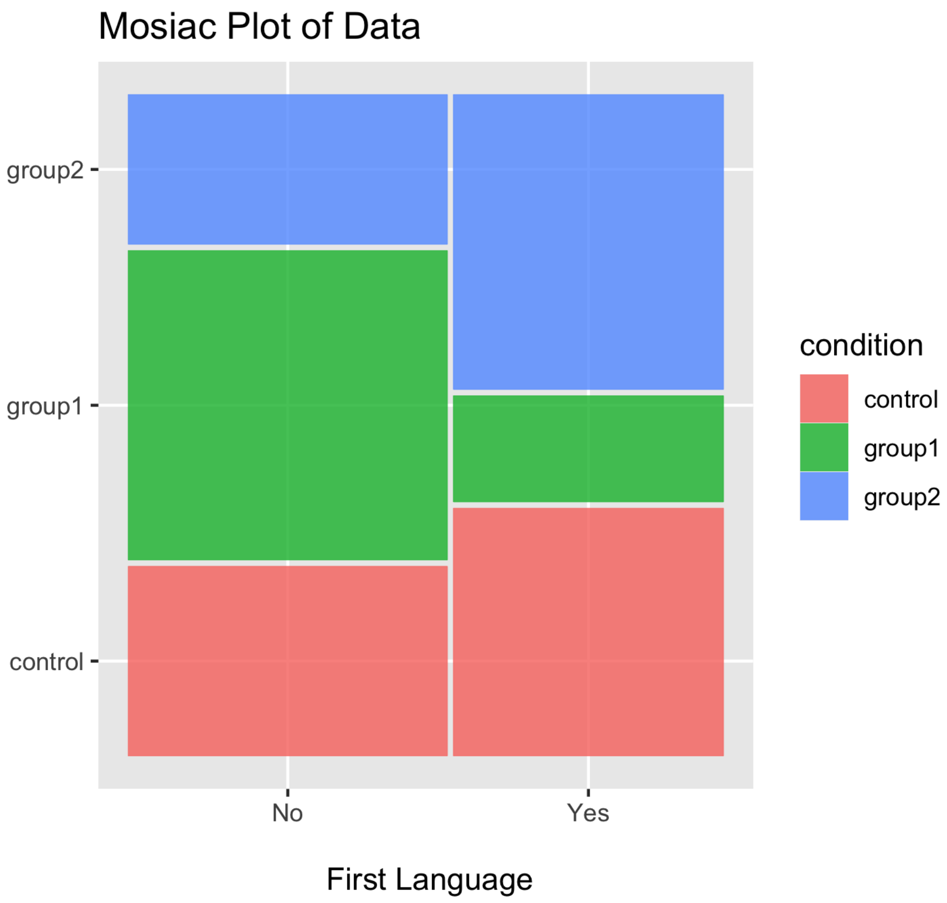
$$H_0 : P_{11} = P_{12}, P_{21} = P_{22}, P_{31} = P_{32}$$
$$H_1 : P_{11} \neq P_{12} | P_{21} \neq P_{22} | P_{31} \neq P_{32}$$

- H_0 says the proportion of each cell in each row are equal.
- H_1 says at least one of these pairs are not equal.

	No	Yes
Control	P11	P12
Group1	P21	P22
Group2	P31	P32

Intuition about the null





Test statistic

- The test statistic looks much the same as the statistic for the GoF test.

$$\chi^2 = \sum_{i=1}^{r} \sum_{i=1}^{c} \frac{(\hat{E}_{ij} - O_{ij})^2}{\hat{E}_{ij}}$$

- What is different?
  - $\sum_{i=1}^{r} \sum_{i=1}^{c}$ simply means sum the quantities for all cells in all rows (r) and columns (c)
  - But why $\hat{E}_{ij}$? Why the hat?

Expected frequencies

- Remember in the GoF test we knew the expected frequencies because we had known proportions and known sample size.
  - Here we do not have that.
- So we have to estimate the expected frequencies from the data.
  - Hence we use $\hat{E}$ to show this is an estimate.

$$\hat{E}_{ij} = \frac{R_i C_j}{N}$$

- Where
  - R_i = the row marginal for a cell i
  - C_i = the column marginal for a cell j
  - N = total sample size
- Here we will show the calculation for one cell (for the cell by cell calculations see the additional material).

Calculation: Controls-No

```
##
##             No Yes Sum
##   control   19  21  40
##   group1    31   9  40
##   group2    15  25  40
##   Sum       65  55 120
```

$$\hat{E_{11}} = \frac{R_1 C_1}{N} = \frac{40 * 65}{120} = \frac{2600}{120} = 21.67$$

$$\frac{(\hat{E_{11}} - O_{11})^2}{\hat{E_{11}}} = \frac{(21.67 - 19)^2}{21.67} = \frac{7.1289}{21.67} = 0.33$$

Null Distribution

- Again, we evaluate the χ^2 test of independence statistic against the χ^2-distribution.
- Here:

$$df = (r-1)(c-1)$$

- Note, r and c are just the number of levels for each categorical variable.
- In our example $(r-1)(c-1) = (3-1)(2-1) = 2 * 1 = 2$
  - Thus using the same α=0.05, we would have the same critical value = 5.99

In R

```
con <- table(exp$condition, exp$lang)
ind_res <- chisq.test(con)
ind_res
```

```
##
##     Pearson's Chi-squared test
##
## data:  con
## X-squared = 13.964, df = 2, p-value = 0.0009286
```

Write up

A χ^2 test of independence was performed to examine whether the distribution of English first language speakers was consistent across experimental conditions (n=120). The relation between these variables was significant (χ^2(2) = 13.96, p <.05). Therefore, we reject the null hypothesis.

Time for a break

For your mid-lecture exercise, please look over the full calculations of the test statistic for this example in the additional slides.

Welcome Back!

Our last recording for this week will look at cell residuals, assumptions, corrections and effect size.

Output

- Here I want to make brief comment about analysis objects.
- The object `ind_res` contains the output of our analysis.
    - This has lots of elements to it.
- We can view and work with these by using the $ sign

```
names(ind_res)
```

```
## [1] "statistic" "parameter" "p.value"   "method"    "data.name" "observed"
## [7] "expected"  "residuals" "stdres"
```

Residuals

- For example, lets look at the residuals.
- The Pearson residuals tell us which cells in the contingency table had the greatest differences.

```
ind_res$residuals
```

```
##
##                   No        Yes
##   control -0.5728919  0.6227992
##   group1   2.0051216 -2.1797970
##   group2  -1.4322297  1.5569979
```

Assumptions

- Sufficiently large N to approximate a normal sampling distribution
    - We saw last semester this actually begins to happen pretty fast.
- Expected and observed cell frequencies are sufficiently large.
    - If either drop below 5, then there is not really enough data.
- Each observation appears in only 1 cell.
    - Data are independent.
    - If data are dependent, we can use a McNemar test.

Yate's correction

- Our χ^2 test only approximates a χ^2 sampling distribution.
- When we have a 2x2 table with df=1, it turns out this approximation is not very good.
    - So for 2x2 tables we apply Yate’s continuity correction.
    - This subtracts 0.5 from each cell deviation.
    - It is the default in R when we have a 2x2 table.

Effect size

- Three possibilities:
  - Phi coefficient (for 2x2 tables)
  - Odds ratios
  - Cramer's V
- We will discuss odds ratios more in year 2, so let's look at Phi and Cramer's V.

Effect size

- The equations for both measures are shown below:

$$Phi = \sqrt{\frac{\chi^2}{N}}$$

$$CramerV = \sqrt{\frac{\chi^2}{N * min(r-1, c-1)}}$$

- Cramer's V generalizes Phi to larger contingency tables.

Cramer's V

- There is no base R calculation for Cramer's V.
- It is included in the `lsr` package for the Navarro book.
- Else we can construct it ourselves.

Cramer's V

```
CV = sqrt(ind_res$statistic /
    (length(exp$ID) *
       (min(length(unique(exp$condition)),
            length(unique(exp$lang))
            ) - 1)))
CV
```

```
## X-squared
## 0.3411211
```

Summary of today

- We have looked at tests for categorical data:

  1. Against a known distribution
  2. As a test of independence.

- We have considered the calculations, inferential tests, and interpretations.

Additional Materials

Full calculations

```
ind_res
```

```
##
##      Pearson's Chi-squared test
##
## data:  con
## X-squared = 13.964, df = 2, p-value = 0.0009286
```

- Let's do all the steps to calculate χ^2 and the exact p-value.

Full calculations

- Let's start with the expected values

$$\hat{E}_{ij} = \frac{R_i C_j}{N}$$

Full calculations

```
##
##            No Yes Sum
##   control  19  21  40
##   group1   31   9  40
##   group2   15  25  40
##   Sum      65  55 120
```

$$\hat{E_{11}} = \frac{R_1 C_1}{N} = \frac{40 * 65}{120} = \frac{2600}{120} = 21.67$$

- As we have the same number of participants in each condition, this is also the expected value for $\hat{E_{21}}$ and $\hat{E_{31}}$

Full calculations

```
##
##            No Yes Sum
##   control  19  21  40
##   group1   31   9  40
##   group2   15  25  40
##   Sum      65  55 120
```

$$\hat{E_{12}} = \frac{R_1 C_2}{N} = \frac{40 * 55}{120} = \frac{2200}{120} = 18.33$$

- As we have the same number of participants in each condition, this is also the expected value for $\hat{E_{22}}$ and $\hat{E_{23}}$

Full calculations

- We can check these against the information in the output to the R analysis

```
ind_res$expected
```

```
##
##                   No      Yes
##   control 21.66667 18.33333
##   group1  21.66667 18.33333
##   group2  21.66667 18.33333
```

Full calculations

- Now, the χ^2

```
##
##              No Yes Sum
##   control    19  21  40
##   group1     31   9  40
##   group2     15  25  40
##   Sum        65  55 120
```

$$\frac{(\hat{E_{11}} - O_{11})^2}{\hat{E_{11}}} = \frac{(21.67 - 19)^2}{21.67} = \frac{7.1289}{21.67} = 0.33$$

Full calculations

- Now, the χ^2

```
##
##            No Yes Sum
##   control  19  21  40
##   group1   31   9  40
##   group2   15  25  40
##   Sum      65  55 120
```

$$\frac{(\hat{E_{21}} - O_{21})^2}{\hat{E_{21}}} = \frac{(21.67 - 31)^2}{21.67} = \frac{87.05}{21.67} = 4.02$$

Full calculations

- Now, the χ^2

```
##
##              No Yes Sum
##   control    19  21  40
##   group1     31   9  40
##   group2     15  25  40
##   Sum        65  55 120
```

$$\frac{(\hat{E_{31}} - O_{31})^2}{\hat{E_{31}}} = \frac{(21.67 - 15)^2}{21.67} = \frac{44.49}{21.67} = 2.05$$

Full calculations

- Now, the χ^2

```
##
##              No Yes Sum
##   control    19  21  40
##   group1     31   9  40
##   group2     15  25  40
##   Sum        65  55 120
```

$$\frac{(\hat{E_{12}} - O_{12})^2}{\hat{E_{12}}} = \frac{(18.33 - 21)^2}{18.33} = \frac{7.1289}{18.33} = 0.39$$

Full calculations

- Now, the χ^2

```
##
##              No Yes Sum
##    control   19  21  40
##    group1    31   9  40
##    group2    15  25  40
##    Sum       65  55 120
```

$$\frac{(\hat{E_{22}} - O_{22})^2}{\hat{E_{22}}} = \frac{(18.33 - 9)^2}{18.33} = \frac{87.05}{18.33} = 4.75$$

Full calculations

- Now, the χ^2

```
##
##               No Yes Sum
##   control     19  21  40
##   group1      31   9  40
##   group2      15  25  40
##   Sum         65  55 120
```

$$\frac{(\hat{E_{32}} - O_{32})^2}{\hat{E_{32}}} = \frac{(18.33 - 25)^2}{18.33} = \frac{44.49}{18.33} = 2.43$$

Full calculations

- Last step is to add them up:

$$\chi^2 = \sum_{i=1}^{r}\sum_{i=1}^{c}\frac{(\hat{E}_{ij} - O_{ij})^2}{\hat{E}_{ij}}$$

```
x2i <- 0.33 + 4.02 + 2.05 + 0.39 + 4.75 + 2.43
x2i
```

```
## [1] 13.97
```

Full calculations

- And check against the R results (tiny bit of rounding error)

```
ind_res
```

```
##
##      Pearson's Chi-squared test
##
## data:  con
## X-squared = 13.964, df = 2, p-value = 0.0009286
```

Full calculations

- And the p-value

```
1 - pchisq(13.964, 2)
```

```
## [1] 0.0009284445
```

Full calculations

- The Pearson's residuals are calculated as:

$$Residual_{ij} = \frac{(E_{ij} - O_{ij})}{\sqrt{E_{ij}}}$$

Full calculations

- So let's do one residual and then look at the output of our analysis:

$$Residual_{11} = \frac{(E_{11} - O_{11})}{\sqrt{E_{11}}} = \frac{(21.67 - 19)}{\sqrt{21.67}} = \frac{2.67}{4.655105} = 0.57$$

```
ind_res$residuals
```

```
##
##                   No        Yes
##   control -0.5728919  0.6227992
##   group1   2.0051216 -2.1797970
##   group2  -1.4322297  1.5569979
```

Full calculations

- Hold on....why is our calculation positive, and the R results negative?
- This is just an interpretation point.
    - In our calculation, we have used $E_{ij} - O_{ij}$
    - If instead we calculate $O_{ij} - E_{ij}$, then we would get the same absolute value but negative.
    - Why not try it.